Analytic Pricing of Quanto CDS

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Abstract

We seek analytic formulae for the pricing of cross-currency credit default swaps (quanto CDS) where the protection payment upon default may be capped in terms of a currency different from that of the debt notional. We assume that the foreign currency and the credit default risk (specified for debt denominated in the *domestic* currency) are potentially correlated with the exchange rate between foreign and domestic currency; and further that this exchange rate may jump by a fixed relative amount in the event of a default. The stochastic credit intensity model is assumed to be Black-Karasinski (lognormal) and the foreign currency interest rate model to be Hull-White (normal). The domestic currency interest rate is assumed to be deterministic, although we could equally assume a stochastic rates model uncorrelated with the other (stochastic) variables.

The solutions are obtained as perturbation expansions which are valid in the limit of the fluctuations of the foreign interest rate and of the credit default intensity from their forward values being small. A Green's function solution is found for the governing PDE which is asymptotically valid under this assumption. This is used to calculate to first order accuracy the price of CDS with a loss cap in a different currency, in particular taking account of the assumed correlations. The solution for foreign currency-denominated debt is compared in the absence of a loss cap with Monte Carlo solutions and found to agree well. An analysis is also performed of the pricing discrepancy introduced by a model incapable of capturing correlation and explicit approximate expressions derived therefor.

1 Introduction

We consider an economy with two currencies, foreign and domestic, and a credit default intensity process associated either with a corporation based in the foreign economy or a foreign sovereign issuer, the CDS market for which operates in the *domestic* currency. We suppose that the foreign interest rate is stochastic but, in the interests of tractability, that the domestic interest rate is deterministic. We also suppose that there is a stochastic exchange rate between the two currencies and further that upon default of the named issuer there is a *decrease* in the value of the foreign currency giving rise to a proportional downward jump in the exchange rate from foreign to domestic currency.

A similar model was first considered by Ehlers & Schönbucher (2006), who took the interest rate and credit intensity models to be affine, viz. in practice either of Hull & White (1990) or of Cox, Ingersoll & Ross (1985) (CIR) type. They did not solve directly the equations they derived but inferred jump levels from market data in the light of their modelling approach. Related work was reported by EL-Mohammadi (2009) who supposed the credit intensity to be lognormally distributed (with no mean reversion) and used his model to price defaultable FX options. A Black & Karasinski (1991) model was first considered in this context by Brigo et al. (2015) in their work reviewing the evidence from the comparative USD and EUR CDS rates for Italian sovereign debt of an implied FX jump in the wake of the euro crisis. Both the previous authors calculated prices using finite difference approaches. In more recent work by Itkin et al. (2017), both the interest rates were taken to be stochastic, governed by a CIR model, the foreign rate allowing of a jump at default as with the FX rate; the credit model was again taken to be Black-Karasinski.

A mathematical model combining the Black-Karasinski credit model of Brigo et al. (2015) with a stochastic interest rate model, as proposed by Ehlers & Schönbucher (2006), was proposed by Turfus (2017) for quanto pricing of defaultable Libor flows with an imposed cap and/or floor. Analytic formulae were derived in this case based on a perturbation expansion approach on the basis of the foreign interest rate and the credit default intensity being small. Effectively the same modelling approach is applied here to the pricing of quanto CDS, with the interest rate model assumed to be Hull-White.

We look to price CDS with the protection payment on a foreign currency-denominated debt capped in terms of a domestic currency amount; likewise for domestic currency-denominated debt capped in terms of a foreign currency amount. We start in section 2 by defining the underlying processes in terms of suitably chosen auxiliary variables and the no-arbitrage conditions they are required to satisfy; we infer therefrom the governing PDE for contingent claims. Given the analytic intractability of this PDE, we define in section 3.1 an asymptotic scaling based on the assumed smallness of the fluctuations of the two short rates: foreign interest rate and credit default intensity. This gives rise to a tractable leading order PDE with the 'intractable' parts isolated as a perturbation; we can then seek solutions as perturbation series. Rather than looking to obtain particular solutions directly, a Green's function for the full PDE is sought as a perturbation expansion, details of which are given in section 3.2. This Green's function is then used to derive asymptotically valid pricing formulae for quanto CDS in section 4. A model risk analysis is carried out in section 5 where the impact of incorrectly ignoring correlation effects is quantified. Finally, conclusions are set out in section 6.

2 Stochastic Modelling

2.1 Definition of Underlying Processes

Our modelling approach is to represent the foreign interest rate and the credit default intensity for a named debt issuer as short rate diffusion processes and the FX rate as a jump diffusion. Specifically, we suppose the foreign interest rate process $r_t : [0, T_m] \to \mathbb{R}$ to be governed by a Hull-White short rate model, with T_m the longest time for which our model is calibrated. We consider a Cox process for default events driven by a credit default intensity process $\lambda_t : [0, T_m] \to \mathbb{R}^+$ governed by a Black-Karasinski (lognormal) short rate model. The exchange rate Z_t from foreign into domestic currency we take to be given by a jump-diffusion process, with a downward jump of a fixed relative amount $k < 0$ occurring at the time of a default of the named issuer. Such jumps are commonly inferred from market data in relation to credit issuers which are perceived to carry systemic risk, particularly but not exclusively in the context of emerging markets, as has been documented by Ehlers & Schönbucher (2006), Brigo et al. (2015) and Itkin et al. (2017). The diffusive processes are all potentially correlated. We assume no correlation of any variables with the domestic interest rate, which justifies our taking it, for computational and notational convenience, to be deterministic.[1] Thus we take the instantaneous domestic interest rate to be given by $r_d(t)$, assumed known from market data. We have in summary

$r_d(t)$: domestic interest rate;

r_t : foreign interest rate;

λ_t : credit default intensity for debt denominated in domestic currency;

Z_t : exchange rate from foreign to domestic currency.

We shall find it convenient to represent r_t and λ_t using auxiliary processes $\tilde{x}_t$ and y_t, respectively, satisfying the following Ornstein-Uhlenbeck equations for $t \geq 0$:

$$d\tilde{x}_t = -\alpha_r(t)\, \tilde{x}_t dt + \sigma_r(t)\, d\tilde{W}_t^1, \tag{2.1}$$

$$dy_t = -\alpha_\lambda(t)\, y_t dt + \sigma_\lambda(t)\, dW_t^2, \tag{2.2}$$

where $\tilde{W}_t^1$ and W_t^2 are Brownian motions under the foreign currency and domestic currency equivalent martingale measures respectively, with

$$\mathrm{corr}(\tilde{W}_t^1, W_t^2) = \rho_{r\lambda}$$

assumed and $\tilde{x}_0 = y_0 = 0$. These auxiliary variables are related to the foreign interest rate r_t and the credit default intensity λ_t, respectively, by

$$r_t = \overline{r}(t) + r^*(t) + \tilde{x}_t, \tag{2.3}$$

$$\lambda_t = (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}(y_t). \tag{2.4}$$

Here $\overline{r}(t)$ is the instantaneous forward rate for foreign currency and $\overline{\lambda}(t)$ the associated credit spread (see (2.18) below), with $\sigma_r(t) > 0$ and $\sigma_\lambda(t) > 0$ their respective volatilities and $\alpha_r(t) > 0$ and $\alpha_\lambda(t) > 0$ their mean reversion rates. Here $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation of a process X_t under the requisite measure. The required form of the configurable functions $r^*(t)$ and $\lambda^*(t)$ is determined by calibration of the model to satisfy the no-arbitrage conditions set out below. The interest rate model defined thus is Hull-White and the respective credit model Black-Karasinski.

By application of the Girsanov theorem, (2.1) can be written equivalently as

$$d\tilde{x}_t = -\alpha_r(t)\, \tilde{x}_t dt - \rho_{rz}\sigma_r(t)\sigma_z(t)dt + \sigma_r(t)\, dW_t^1, \tag{2.5}$$

[1] In the absence of correlation between the domestic interest rate and any of the other three stochastic processes, stochasticity of the domestic interest rates can be shown to have no impact on the prices we calculate.

with W_t^1 a Brownian motion under the *d_ domestic* currency equivalent martingale measure. This has solution subject to $\tilde{x}_0 = 0$ given by

$$\tilde{x}_t = -I_{rz}(0,t) + \int_0^t \phi_r(s,t)\sigma_r(s)dW_s^1 \tag{2.6}$$

where

$$\phi_r(s,t) := e^{-\int_s^t \alpha_r(u)du}, \tag{2.7}$$

$$I_{rz}(t_1,t_2) := \rho_{rz}\int_{t_1}^{t_2} \phi_r(u,t_2)\sigma_r(u)\sigma_z(u)du. \tag{2.8}$$

Finally we suppose, following Ehlers & Schönbucher (2006), that the exchange rate Z_t from foreign to domestic currency is governed by

$$\frac{dZ_t}{Z_t} = (r_d(t) - r_t - k\lambda_t)\,dt + \sigma_z(t)\,dW_t^3 + kdn_t. \tag{2.9}$$

where dW_t^3 is a Brownian motion, n_t a Cox process with intensity λ_t under the domestic currency equivalent martingale measure and $k \in (-1,\infty)$ is the deterministic jump at default, typically < 0. We further suppose

$$\mathrm{corr}(W_t^1, W_t^3) = \rho_{rz},$$
$$\mathrm{corr}(W_t^2, W_t^3) = \rho_{\lambda z}.$$

It will be convenient to express Z_t also through an auxiliary variable, z_t, defined such that

$$Z_t = F(t)\mathcal{E}(z_t), \tag{2.10}$$

where

$$F(t) = Z_0 e^{\int_0^t (r_d(s)-\overline{r}(s))ds}, \tag{2.11}$$

$$\mathcal{E}(z_t) = \exp\left(z_t - \tfrac{1}{2}I_{zz}(t) - k\int_0^t \overline{\lambda}(s)ds\right), \tag{2.12}$$

$$I_{zz}(t_1,t_2) = \int_{t_1}^{t_2} \sigma_z^2(u)\,du, \tag{2.13}$$

and Z_0 is assumed known, whence we infer

$$dz_t = -\left(r_t - \overline{r}(t) + k(\lambda_t - \overline{\lambda}(t))\right)dt + \sigma_z(t)\,dW_t^3 + kdn_t. \tag{2.14}$$

with $z_0 = 0$.

The no-arbitrage conditions

The formal no-arbitrage constraints which determine the functions $r^*(t)$ and $\lambda^*(t)$ are as follows. First, by considering a risk-free foreign currency cash flow at time t, we deduce

$$\mathbb{E}^f\left[e^{-\int_0^t r_s\,ds}\right] = D(0,t), \tag{2.15}$$

for $0 < t \le T_m$, where T_m is the longest maturity date for which the model is calibrated, and

$$D(t_1,t_2) = e^{-\int_{t_1}^{t_2} \overline{r}(s)\,ds} \tag{2.16}$$

is the t_1-forward price of the t_2-maturity zero coupon bond denominated in foreign currency. Likewise, considering a *risky* domestic currency cash flow, we have

$$\mathbb{E}^d\left[e^{-\int_0^t (r_d(s)+\lambda_s)\,ds}\right] = B(0,t), \tag{2.17}$$

where

$$B(t_1,t_2) = e^{-\int_{t_1}^{t_2} (r_d(s)+\overline{\lambda}(s))\,ds} \tag{2.18}$$

is the price of a *risky* domestic currency cash flow. Here $\mathbb{E}^d[\cdot]$ and $\mathbb{E}^f[\cdot]$ represent expectations under the domestic and foreign currency martingale measures respectively. We shall assume the bond prices can be ascertained at the initial time $t = 0$ from the market, whence we can view (2.16) and (2.18) as implicitly *defining* $\overline{r}(t)$ and $\overline{\lambda}(t)$.

2.2 Derivation of Governing PDE

For future notational convenience we replace $\tilde{x}_t$ by a new variable x_t defined by

$$x_t := \tilde{x}_t + I_{rz}(0,t), \tag{2.19}$$

in terms of which (2.3) and (2.5) become

$$r_t = \overline{r}(t) + r^*(t) - I_{rz}(0,t) + x_t, \tag{2.20}$$

$$dx_t = -\alpha_r(t)\,x_t dt + \sigma_r(t)\,dW_t^1. \tag{2.21}$$

We also introduce the convenient shorthand notation that, for a process X_t and $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{E}_x(f(t)X_t) := \mathcal{E}(f(t)X_t)|_{X_t=x},$$

making use of which we can write $r_t = r(x_t,t)$ and $\lambda_t = \lambda(y_t,t)$, where

$$r(x,t) := \overline{r}(t) + r^*(t) - I_{rz}(0,t) + x, \tag{2.22}$$

$$\lambda(y,t) := (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}_y(y_t), \tag{2.23}$$

We consider in the first instance the general problem of pricing a cash security with maturity T whose payoff depends on x_T and z_T. Writing the price of the security in domestic currency at time $t \in [0,T]$ as $f_t^T = f(x_t,y_t,z_t,t)$, we can infer by application of the Feynman-Kac theorem to (2.2), (2.14) and (2.21) in the standard manner that the function $f(x,y,z,t)$ satisfies the following backward diffusion equation:

$$\left(\frac{\partial}{\partial t} + \hat{\mathcal{L}}\right) f(x,y,z,t) = 0, \tag{2.24}$$

where

$$\begin{aligned}\hat{\mathcal{L}} :=& -\alpha_r(t)x\frac{\partial}{\partial x} - \alpha_\lambda(t)y\frac{\partial}{\partial y} - (r(x,t) - \overline{r}(t) + k(\lambda(y,t) - \overline{\lambda}(t)))\frac{\partial}{\partial z} - (\lambda(y,t) + r_d(t))\cdot \\ &+ \frac{1}{2}\left(\sigma_r^2(t)\frac{\partial^2}{\partial x^2} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial y^2} + \sigma_z^2(t)\frac{\partial^2}{\partial z^2} + 2\rho_{r\lambda}\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2}{\partial x\partial y} + 2\rho_{rz}\sigma_r(t)\sigma_z(t)\frac{\partial}{\partial x\partial z}\right. \\ &\left. + 2\rho_{\lambda z}\sigma_\lambda(t)\sigma_z(t)\frac{\partial^2}{\partial y\partial z}\right),\end{aligned} \tag{2.25}$$

with in general $f_T^T = P(x_T,z_T)$ for some payoff function $P : \mathbb{R}^2 \to \mathbb{R}$. Equally we can consider a default payoff $P_{\text{def}}(x_\tau,z_\tau,\tau)$ at some stopping time τ representing the time of default within our proposed framework with $P_{\text{def}} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$. The domain of definition of our solution will then be $t \in [0,T\wedge\tau]$, where the binary operator $\wedge$ denotes a minimum. In the absence of closed form solutions to (2.24) and guided by the work of Hagan et al. (2015) and Turfus (2017), we propose a perturbation expansion approach as follows.

3 Asymptotic Modelling

3.1 Perturbation Operator

We proceed by defining new functions:

$$\begin{aligned} h(x,t,t_1) &:= r^*(t_1) + \phi_r(t,t_1)(x - I_{rz}(0,t)), \\ g(y,t,t_1) &:= \left(\overline{\lambda}(t_1) + \lambda^*(t_1)\right) \mathcal{E}_y(\phi_\lambda(t,t_1)y_t) - \overline{\lambda}(t_1), \end{aligned} \qquad t_1 \geq t \geq 0 \tag{3.1}$$

with

$$\phi_\lambda(s,t) := e^{-\int_s^t \alpha_\lambda(u)du}, \tag{3.2}$$

and further

$$\begin{aligned} h(x,t) &:= h(x,t,t) = r(x,t) - \overline{r}(t), \\ g(y,t) &:= g(y,t,t) = \lambda(y,t) - \overline{\lambda}(t), \end{aligned} \tag{3.3}$$

in terms of which we can rewrite (2.24) and (2.25) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} + \mathcal{H} + \mathcal{G}\right) f(x,y,z,t) = 0, \tag{3.4}$$

where

$$\begin{aligned} \mathcal{L} :=& - \alpha_r(t)x\frac{\partial}{\partial x} - \alpha_\lambda(t)y\frac{\partial}{\partial y} + \frac{1}{2}\bigg(\sigma_r^2(t)\frac{\partial^2}{\partial x^2} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial y^2} + \sigma_z^2(t)\frac{\partial^2}{\partial z^2} + 2\rho_{r\lambda}\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2}{\partial x \partial y} \\ & + 2\rho_{rz}\sigma_r(t)\sigma_z(t)\frac{\partial^2}{\partial x \partial z} + 2\rho_{\lambda z}\sigma_\lambda(t)\sigma_z(t)\frac{\partial^2}{\partial y \partial z}\bigg) - (\overline{\lambda}(t) + r_d(t))\cdot, \end{aligned} \tag{3.5}$$

$$\mathcal{H} := - h(x,t)\frac{\partial}{\partial z}, \tag{3.6}$$

$$\mathcal{G} := - g(y,t)\left(1 + k\frac{\partial}{\partial z}\right). \tag{3.7}$$

We suppose that the operators $\mathcal{H}$ and $\mathcal{G}$ represent small perturbations of $\mathcal{O}(\epsilon_r)$ and $\mathcal{O}(\epsilon_\lambda)$ magnitude, respectively, and seek to solve (3.4) in the limit as $\epsilon_r, \epsilon_\lambda \to 0$. Formally, this can be achieved by defining the asymptotically small parameters through the following averaged representations of $h(\cdot)$ and $g(\cdot)$:

$$\epsilon_r^2 := \frac{1}{T_m}\int_0^{T_m} \mathrm{var}\left[\int_t^{T_m} h(x_t,t,u)\,du\right] dt, \tag{3.8}$$

$$\epsilon_\lambda^2 := \frac{1}{T_m}\int_0^{T_m} \mathrm{var}\left[\int_t^{T_m} g(y_t,t,u)\,du\right] dt. \tag{3.9}$$

The former can be made small by taking $\sigma_r(t)$ small enough, the latter by taking $\overline{\lambda}(t)$ and/or $\sigma_\lambda(t)$ small enough. In other words, we require that the stochastic fluctuations of the foreign interest rate and the credit intensity away from their respective forward rates should on average be small in absolute terms.

3.2 Green's Function

We seek a Green's function solution for (3.4) as a joint power series in ϵ_r and ϵ_λ. The derivation has been presented in Turfus (2017), albeit using different notation and slightly different assumptions. We

present below the main conclusions using the notation defined above. The result can formally be written asymptotically as

$$G(x,y,z,t;\xi,\eta,\zeta,v) = B(t,v)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} G_{i,j}(x,y,z,t;\xi,\eta,\zeta,v), \tag{3.10}$$

where $G_{i,j}(\cdot) = \mathcal{O}({\epsilon_r}^i{\epsilon_\lambda}^j)$. In practice we restrict our attention to first order terms, viz. those for which $i+j \leq 1$. We will in all cases be interested in so-called free-boundary Green's function solutions which tend to zero as $x, y \to \pm\infty$. The leading order Green's function solution subject to these conditions is well known. It is given by:

$$G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) = \frac{\partial^3}{\partial\xi\partial\eta\partial\zeta} N_3(\xi - \phi_r(t,v)x, \eta - \phi_\lambda(t,v)y, \zeta - z; R(t,v)), \quad t < v \tag{3.11}$$

where $N_3(x,y,z;R(t,v))$ is a trivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t,v) = \begin{pmatrix} I_{rr}(t,v) & I_{r\lambda}(t,v) & I_{rz}(t,v) \\ I_{r\lambda}(t,v) & I_{\lambda\lambda}(t,v) & I_{\lambda z}(t,v) \\ I_{rz}(t,v) & I_{\lambda z}(t,v) & I_{zz}(t,v) \end{pmatrix} \tag{3.12}$$

with

$$I_{rr}(t_1,t_2) = \int_{t_1}^{t_2} \phi_r^2(u,t_2)\sigma_r^2(u)\,du, \tag{3.13}$$

$$I_{\lambda\lambda}(t_1,t_2) = \int_{t_1}^{t_2} \phi_\lambda^2(u,t_2)\sigma_\lambda^2(u)\,du, \tag{3.14}$$

$$I_{r\lambda}(t_1,t_2) = \rho_{r\lambda}\int_{t_1}^{t_2} \phi_r(u,t_2)\phi_\lambda(u,t_2)\sigma_r(u)\sigma_\lambda(u)\,du, \tag{3.15}$$

$$I_{\lambda z}(t_1,t_2) = \rho_{\lambda z}\int_{t_1}^{t_2} \phi_\lambda(u,t_2)\sigma_\lambda(u)\sigma_z(u)\,du \tag{3.16}$$

and $I_{rz}(\cdot)$ and $I_{zz}(\cdot)$ given by (2.8) and (2.13), respectively. Note that $I_{rr}(0,t)$ and $I_{\lambda\lambda}(0,t)$ constitute the quadratic variations of r_t and λ_t, respectively.

Following Turfus (2017) and Turfus (2018), we obtain at first order

$$G_{1,0}(x,y,z,t;\xi,\eta,\zeta,v) = -\left(B^*(t,v)(x - I_{rz}(0,t)) + \frac{I^*(t,v)}{\phi_r(t,v)}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z}G_{0,0}\left(x,y,z,t;\xi,\eta,\zeta,v\right) \tag{3.17}$$

and

$$G_{0,1}(x,y,z,t;\xi,\eta,\zeta,v) = -\int_t^v \overline{\lambda}(u)\left(\mathcal{E}_y(\phi_\lambda(t,u)y_t)\mathcal{M}_{t,u} - 1\right)\left(1 + k\frac{\partial}{\partial z}\right)G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\,du, \tag{3.18}$$

where we have defined

$$B^*(t,v) := \int_t^v \phi_r(t,u)\,du, \tag{3.19}$$

$$I^*(t,v) := \int_t^v \phi_r(u,v)I_{rr}(t,u)\,du, \tag{3.20}$$

$$\mathcal{M}_{t_1,t_2}G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) := G_{0,0}\left(x, y + \frac{I_{\lambda\lambda}(t_1,t_2)}{\phi_\lambda(t,t_2)}, z + \frac{I_{\lambda z}(t_1,t_2)}{\phi_\lambda(t,t_2)}, t;\xi,\eta,\zeta,v\right) \tag{3.21}$$

Turfus (2017) further infers that satisfaction of the no arbitrage conditions (2.15) and (2.17) to second order accuracy requires

$$r^*(t) \sim \gamma_{2,0}^*(t) = I^*(0,t) \tag{3.22}$$

$$\lambda^*(t) \sim \gamma_{0,2}^*(t) = \overline{\lambda}(t)\int_0^t \overline{\lambda}(u)\left(e^{\phi_\lambda(u,t)I_{\lambda\lambda}(0,u)} - 1\right)du, \tag{3.23}$$

with $\gamma_{i,j} = \mathcal{O}({\epsilon_r}^i {\epsilon_\lambda}^j)$. Substituting (3.11), (3.17) and (3.18) into (3.10) then gives the required Green's function to first order accuracy.

Since the differential operators appearing in the Green's function apply only to z and not to ξ, η or ζ, they do not influence the applicability of the Green's function to the requisite payoff function, which consequently has no smoothness conditions imposed other than integrability, say by its being piecewise continuous and exponentially bounded.

4 Pricing Formulae

4.1 Coupon Leg

We suppose that the CDS holder makes fixed payments in the foreign currency conditional on no default at a coupon rate c, based on a payment schedule $0 < t_1 < t_2 < \ldots < t_n$.[2] Thus the value of the payment at time $t_i > 0$ is assumed to be given in terms of domestic currency by $Z_{t_i} c\Delta_i$, where Δ_i is the year fraction between t_{i-1} and t_i. Thus the payoff function at time t_i can be written, using (2.10), as

$$P_i(z) F(t_i) \mathcal{E}_z(z_{t_i}) c\Delta_i. \tag{4.1}$$

On this basis we can express the value at time $t < t_i$ of the ith coupon payment as follows:

$$F_i(x,y,z,t) = \iiint_{\mathbb{R}^3} G(x,y,z,t;\xi,\eta,\zeta,t_i) P_i(\zeta)\, d\xi\, d\eta\, d\zeta. \tag{4.2}$$

Using our first order representation of $G(\cdot)$ from the previous section, we deduce straightforwardly:

$$F_i(x,y,z,t) \sim B(t,t_i) F(t_i) e^{-k\int_t^{t_i} \overline{\lambda}(s)ds} c\Delta_i \mathcal{E}_z(z_t) \left(1 - \int_t^{t_i} (h(x,t,u) + (1+k) g(y,t,u))\, du\right) \tag{4.3}$$

In particular, we find

$$F_i(0,0,0,0) \sim B(0,t_i) F(t_i) e^{-k\int_0^{t_i} \overline{\lambda}(s)ds} c\Delta_i \left(1 - (1+k) \int_0^{t_i} \overline{\lambda}(u) \left(e^{I_{\lambda z}(0,t_i)} - 1\right) du\right), \tag{4.4}$$

using which we can write the PV of the coupon leg as

$$V_{\text{coupon}} \sim c \sum_{i=1}^{n} B(0,t_i) F(t_i) e^{-k\int_0^{t_i} \overline{\lambda}(s)ds} \Delta_i \left(1 - (1+k) \int_0^{t_i} \overline{\lambda}(u) \left(e^{I_{\lambda z}(0,t_i)} - 1\right) du\right). \tag{4.5}$$

We infer that a positive FX-credit correlation will act so as to decrease the PV of the coupon leg, whereas the impact of a negative jump $k < 0$ will be such as to cause a small increase. Note the difference with the PV of the protection leg in the following section which is seen to increase in both these circumstances

4.2 Protection Leg

We consider two types of protection leg, both priced using a default curve calibrated to the domestic currency CDS market. In the first place we suppose that the debt is denominated in the foreign currency and may be capped by a domestic currency-denominated amount. Secondly, we consider the case where the debt is denominated in the domestic currency but the loss is capped at a foreign currency-denominated amount. This latter situation is relevant when a financial institution sells CDS protection to a foreign client collateralised in the client's currency: if the loss on default exceeds the collateral amount, there will be a gap loss. We will refer to the first of these cases as quanto protection and to the second as quanto loss cap.

[2] We could also consider payments in the domestic currency but the pricing thereof is independent of the stochastic modelling so needs no further comment here.

4.2.1 Quanto protection

For a protection leg with maturity $T = t_n$, we write the value at time t of the payment that the CDS holder receives contingent on default in $[t, T]$ as $Q(x, y, z, t)$. There is no payoff at maturity, so the final condition is $Q(x, y, z, T) = 0$. We take the assumed recovery level on the underlying debt to be R and the cap in domestic currency on the recovery payment per unit of foreign notional to be K. Taking account also of the fact that the FX rate jumps by a relative amount k at default time τ, we can write the payment at default as

$$P_{\text{def}}(z, \tau) = \min\{(1 - R)(1 + k)F(\tau)\mathcal{E}_z(z_{\tau^-}), K\}, \tag{4.6}$$

where, by the use of τ^-, we indicate that the limit is taken of the FX rate as $t \uparrow \tau$, viz. prior to the jump occurring. Noting also that the instantaneous default intensity at time t is given by $\overline{\lambda}(t) + g(y, t)$, we conclude that $Q(x, y, z, t)$ satisfies

$$\left(\frac{\partial}{\partial t} + \mathcal{L} + \mathcal{H} + \mathcal{G}\right) Q(x, y, z, t) = -(\overline{\lambda}(t) + g(y, t))P_{\text{def}}(z, t), \tag{4.7}$$

for $t < T \wedge \tau$. We observe here that $\mathcal{H}$ is $\mathcal{O}(\epsilon_r)$ and $\mathcal{G}$ and $g(y, t)$ are $\mathcal{O}(\epsilon_\lambda)$. Consequently we pose for our (first order) solution:

$$Q(x, y, z, t) = \sum_{i+j \le 1}\sum Q_{i,j}(x, y, z, t), \tag{4.8}$$

where $Q_{i,j} = \mathcal{O}({\epsilon_r}^i {\epsilon_\lambda}^j)$. The details of the calculation up to first order (including also the most important second order terms) are presented in Appendix A. We state here only the conclusion that

$$\begin{aligned}
Q(x, y, z, t) \sim \int_t^T & B(t, v)(\overline{\lambda}(v) + \gamma^*_{0,2}(v))\mathcal{E}_y(\phi_\lambda(t, v)y_t) \\
& \qquad (M(v)\mathcal{E}_z(z_t + I_{\lambda z}(t, v))N(-d_1(z, t, v)) + KN(d_2(z, t, v)))\, dv \\
- \int_t^T & B(t, v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t, v)y_t)M(v)\mathcal{E}_z(z_t + I_{\lambda z}(t, v))B^*(t, v)(x - I_{r\lambda}(0, t)) \\
& \qquad N(-d_1(z, t, v))\, dv \\
- \int_t^T & B(t, v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t, v)y_t)M(v)\mathcal{E}_z(z_t + I_{\lambda z}(t, v)) \left(1 + k + k\frac{\partial}{\partial z}\right) N(-d_1(z, t, v)) \\
& \qquad \int_t^v \overline{\lambda}(u) \left(\mathcal{E}_y(\phi_\lambda(t, u)y_t)e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(t,u)+I_{\lambda z}(t,u))} - 1\right) du\, dv \\
- K \int_t^T & B(t, v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t, v)y_t) \left(1 + k\frac{\partial}{\partial z}\right) N(d_2(z, t, v)) \\
& \qquad \int_t^v \overline{\lambda}(u) \left(\mathcal{E}_y(\phi_\lambda(t, u)y_t)e^{\phi_\lambda(u,v)I_{\lambda\lambda}(t,u)} - 1\right) du\, dv.
\end{aligned} \tag{4.9}$$

with errors $= \mathcal{O}(\epsilon_r^2 + \epsilon_\lambda^2)$, where $M(\cdot)$ is given by (A.5) and the $d_i(\cdot)$ by (A.6) and (A.7). Finally, setting $x = y = t = 0$, we obtain

$$
\begin{aligned}
V_{\text{protection}} \sim & \int_0^T B(0,v)\overline{\lambda}(v)\left(M(v)e^{I_{\lambda z}(0,v)}N(-d_1(v)) + KN(d_2(v))\right)dv \\
& - \int_0^T B(0,v)\overline{\lambda}(v)M(v)e^{I_{\lambda z}(0,v)}\left((1+k)N(-d_1(v)) - k\frac{N'(-d_1(v))}{\sqrt{I_{zz}(0,v)}}\right) \\
& \qquad \int_0^v \overline{\lambda}(u)\left(e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(0,u)+I_{\lambda z}(0,u))} - 1\right)du\,dv \\
& - K\int_0^T B(0,v)\overline{\lambda}(v)\left(N(d_2(v)) + k\frac{N'(d_2(v))}{\sqrt{I_{zz}(0,v)}}\right)\int_0^v \overline{\lambda}(u)\left(e^{\phi_\lambda(u,v)I_{\lambda\lambda}(0,u)} - 1\right)du\,dv \\
& + \int_0^T B(0,v)\overline{\lambda}(v)\left(M(v)e^{I_{\lambda z}(0,v)}N(-d_1(v)) + KN(d_2(v))\right)\int_0^v \overline{\lambda}(u)\left(e^{\phi_\lambda(u,v)I_{\lambda\lambda}(0,u)} - 1\right)du\,dv,
\end{aligned}
\tag{4.10}
$$

where we have used $d_i(v)$ as a shorthand for $d_i(0,0,v)$. Simplifying we obtain[3]

$$
\begin{aligned}
V_{\text{protection}} \sim & \int_0^T B(0,v)\overline{\lambda}(v)\left(M(v)e^{I_{\lambda z}(0,v)}N(-d_1(v)) + KN(d_2(v))\right)dv \\
& - \int_0^T B(0,v)\overline{\lambda}(v)M(v)e^{I_{\lambda z}(0,v))}\left(N(-d_1(v)) - k\frac{N'(-d_1(v))}{\sqrt{I_{zz}(0,v)}}\right) \\
& \qquad \int_0^v \overline{\lambda}(u)e^{\phi_\lambda(u,v)I_{\lambda\lambda}(0,u)}\left(e^{\phi_\lambda(u,v)I_{\lambda z}(0,u)} - 1\right)du\,dv \\
& - k\int_0^T B(0,v)\overline{\lambda}(v)M(v)e^{I_{\lambda z}(0,v)}N(-d_1(v))\int_0^v \overline{\lambda}(u)\left(e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(0,u)+I_{\lambda z}(0,u))} - 1\right)du\,dv.
\end{aligned}
\tag{4.11}
$$

This result can be interpreted as follows. In the absence of stochastic credit intensity, the cost of protection will be given by the first line with $I_{\lambda z}(\cdot) \equiv 0$. The leading order impact of an increase in the FX-credit correlation will be to increase the magnitude of all terms. However, the impact in the first term will obviously be much larger, whence the result will be a net *increase* in the value of protection. Similarly, the impact of a downwards jump in the FX rate at default is to decrease the value of protection (through the agency of $M(\cdot)$). Stochastic foreign rates notably have no impact at this level of approximation. It will be noted that terms involving $I_{\lambda\lambda}(\cdot)$ appear which are ostensibly second order. A pure first order expansion can be deduced from (4.11) by setting $I_{\lambda\lambda}(\cdot) = 0$. We argue below that doing so will slightly diminish the quality of results obtained, particularly for larger credit spreads.

In the limit as $K \to \infty$, (4.11) approaches the plain quanto CDS limit:

$$
\begin{aligned}
V^{\text{CDS}}_{\text{protection}} = & \int_0^T M(v)B(0,v)\overline{\lambda}(v)e^{I_{\lambda z}(0,v)} \\
& \left(1 - \int_0^v \overline{\lambda}(u)\left((1+k)e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(0,u)+I_{\lambda z}(0,u))} - e^{\phi_\lambda(u,v)I_{\lambda\lambda}(0,u)} - k\right)du\right)dv.
\end{aligned}
\tag{4.12}
$$

Notice that, even in the absence of FX-credit correlation, the price maintains a dependence on the stochastic credit, other than in the trivial case where $k = 0$ and there is no quanto effect.

The impact of the loss cap K on the cost of protection calculated using (4.11) is illustrated in Fig. 1 against the result of a Monte Carlo calculation performed without a loss cap on a 5y quanto CDS. The

[3] We use here the identity that $M(v)e^{I_{\lambda z}(0,v)}N'(-d_1(v)) = KN'(d_2(v))$.

notional was 100 M JPY and the trade was priced in USD with a JPY/USD exchange rate of 113.58. The 5y fair CDS premium was 32 bp, with $\rho_{\lambda z} = -0.4$ and $k = -0.1$. As expected, the result increases linearly at first with the cap level, flattening off subsequently as the influence of the cap diminishes.

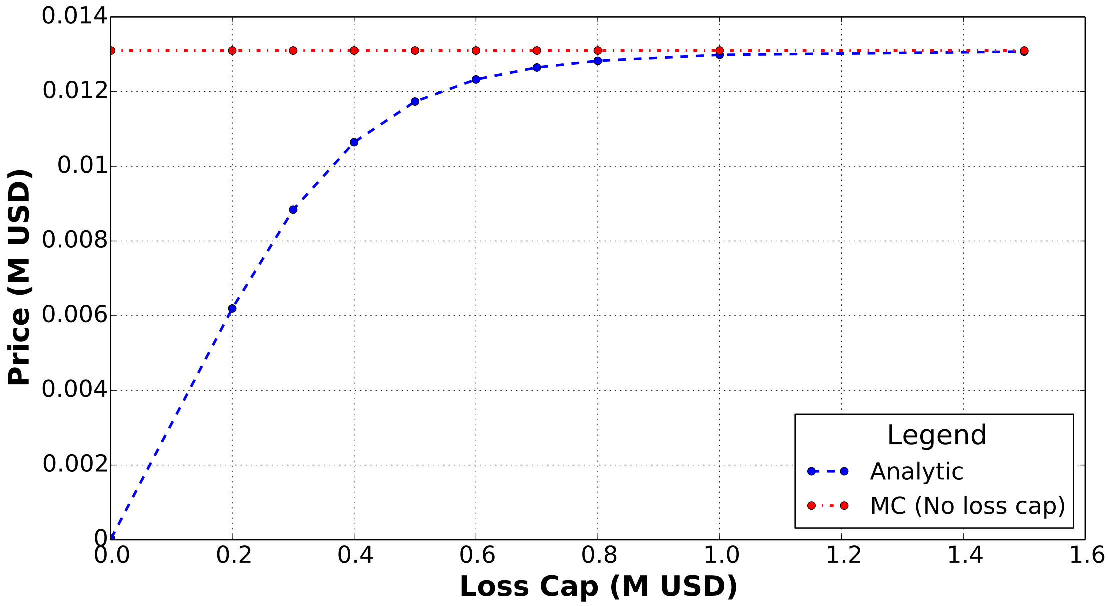

Figure 1: Dependence of cost of quanto protection on domestic currency loss cap K.

4.2.2 Quanto loss cap

We next suppose instead that the debt is denominated in *domestic* currency but the protection payment capped at a *foreign* currency amount K_F. In this case we can write the payment at default as

$$P_{\text{def}}(z,\tau) = \min\{1-R, K_F(1+k)F(\tau)\mathcal{E}_z(z_{\tau^-})\}. \tag{4.13}$$

The calculation is similar to that presented in Appendix A. To the same level of accuracy we obtain

$$\begin{aligned} V^{K_F}_{\text{protection}} \sim & \int_0^T B(0,v)\overline{\lambda}(v)\left((1-R)N(d_2^*(v)) + M^*(v)e^{I_{\lambda z}(0,v)}N(-d_1^*(v))\right)dv \\ & - \int_0^T B(0,v)\overline{\lambda}(v)M^*(v)e^{I_{\lambda z}(0,v))}\left(N(-d_1^*(v)) - k\frac{N'(-d_1^*(v))}{\sqrt{I_{zz}(0,v)}}\right) \\ & \qquad \int_0^v \overline{\lambda}(u)e^{\phi_\lambda(u,v)I_{\lambda\lambda}(0,u)}\left(e^{\phi_\lambda(u,v)I_{\lambda z}(0,u)}-1\right)du\,dv \\ & - k\int_0^T B(0,v)\overline{\lambda}(v)M^*(v)e^{I_{\lambda z}(0,v)}N(-d_1^*(v))\int_0^v \overline{\lambda}(u)\left(e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(0,u)+I_{\lambda z}(0,u))}-1\right)du\,dv, \end{aligned} \tag{4.14}$$

with

$$M^*(v) := K_F(1+k)F(v)e^{-k\int_0^v \overline{\lambda}(s)ds} \tag{4.15}$$

$$\begin{aligned} d_2^*(v) &:= \frac{\ln M^*(v) - \ln(1-R) - \frac{1}{2}I_{zz}(0,v) + I_{\lambda z}(0,v)}{\sqrt{I_{zz}(0,v)}}, \\ d_1^*(v) &:= d_2^*(v) + \sqrt{I_{zz}(0,v)}. \end{aligned} \tag{4.16}$$

This result can be interpreted as follows. In the absence of stochastic credit intensity, the cost of protection will be given by the first line with $I_{\lambda z}(\cdot) \equiv 0$. As previously, the leading order impact of an increase in the FX-credit correlation will be a net *increase* in the value of protection, although in this case the impact vanishes in the limit as $K_F \to \infty$. Again the impact of a downwards jump in the FX rate at default is to reduce the value of protection and stochastic foreign rates have no impact at this level of approximation. In the limit as $K_F \to \infty$, the plain (non-quanto) CDS result is recovered.

The impact of the loss cap K_F on the cost of protection calculated using (4.14) is illustrated in Fig. 2 against the result of a Monte Carlo calculation performed without a loss cap on a 5y quanto CDS. The notional was in this case 100 M USD and the other trade (and market) characteristics as before. As can be seen, the results and the agreement are qualitatively similar to those obtained in the previous case.

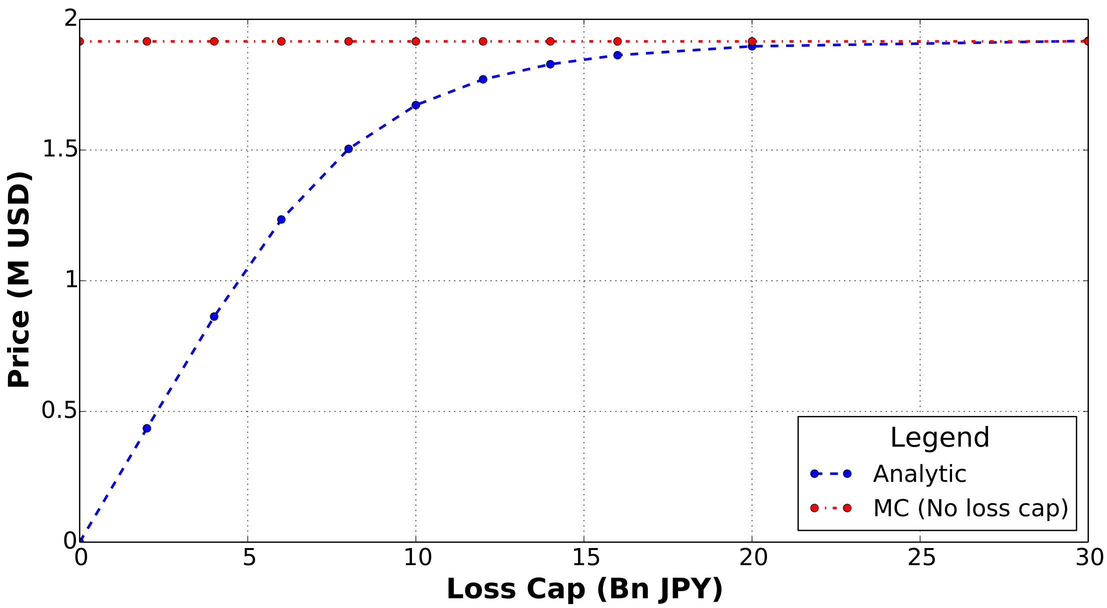

Figure 2: Dependence of cost of protection with quanto loss cap on foreign currency loss cap K_F.

5 Quantifying Correlation Risk

It is of interest to ask, in view of the fact that CDS are often priced with deterministic models, viz. non-stochastic rates and credit, what the formulae derived above tell us about the potential impact of correlation on quanto CDS pricing with capped loss payment. We consider first the accuracy of (4.12) in particular by comparison with Monte Carlo solution of the same problem.

5.1 Comparison with Monte Carlo Simulation

Comparison of results obtained from (4.12) with Monte Carlo simulations for the quanto CDS considered in Fig. 1 above is made in Fig. 3 for various representative values of $\rho_{\lambda z}$. As can be seen, although the impact of correlation can be substantial, the agreement is excellent, barely exceeding 1 bp of notional, particularly when the correlation is negative, as is usually the case. The impact for the same CDS of varying the jump magnitude k with the correlation fixed at -40% is illustrated in Fig. 4: the impact is again substantial, the agreement similarly good.

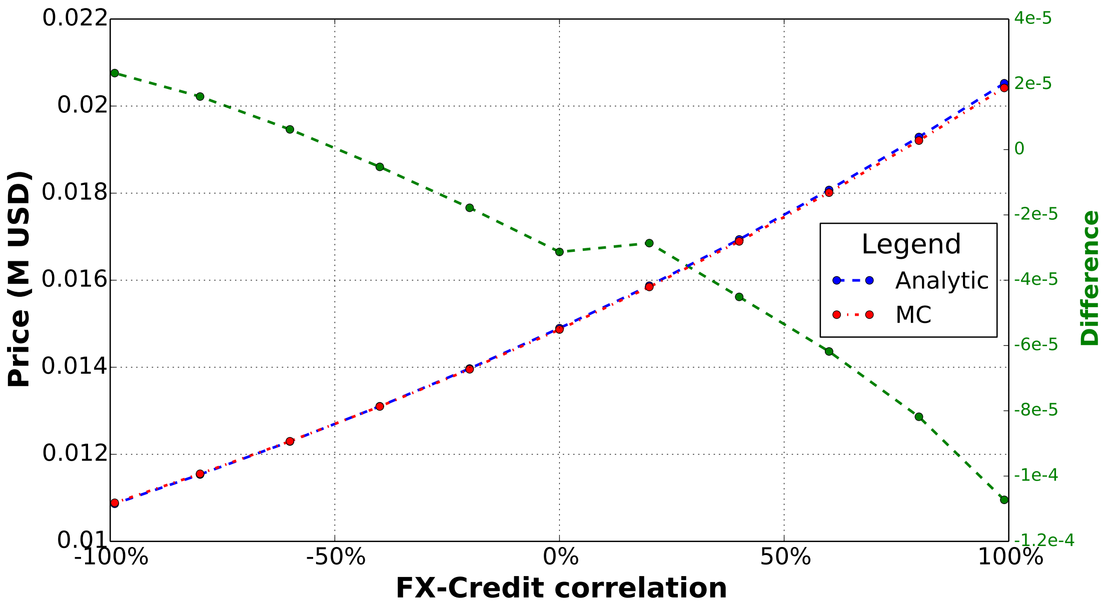

Figure 3: Dependence of cost of protection for quanto CDS on credit-FX correlation.

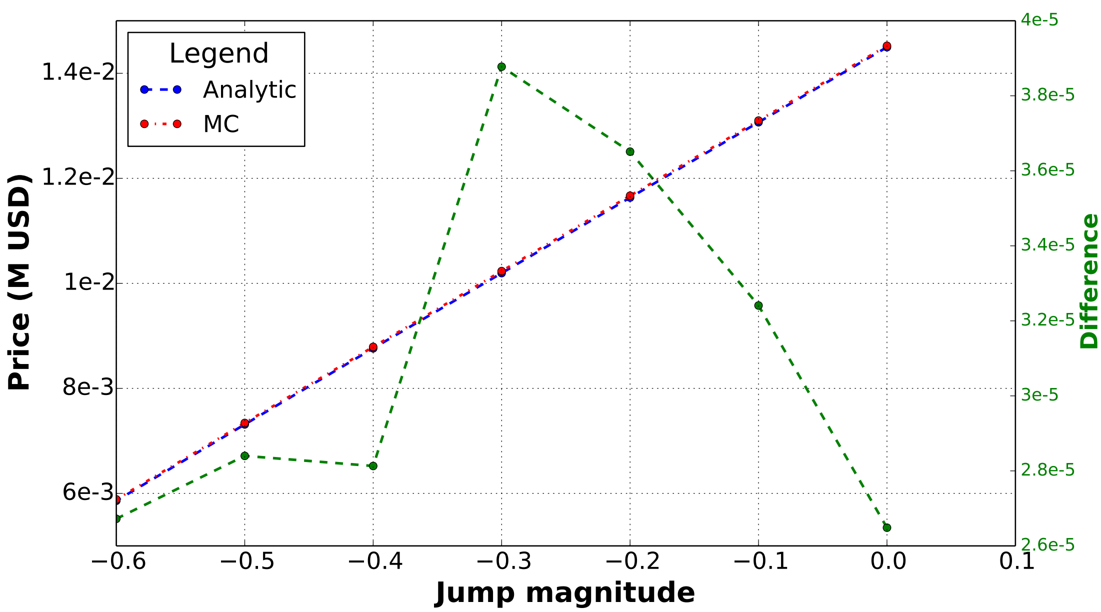

Figure 4: Dependence of cost of protection for quanto CDS on jump magnitude k.

We also consider the impact of increasing the credit risk by bumping the credit intensity curve up by a fixed amount. The results are shown in Fig. 5 for bumps of up to 1000 bp. As can be seen, the discrepancy between the analytic approximation and the Monte Carlo calculation remains surprisingly good, with the relative error actually decreasing for higher credit spreads It is worth noting that, when the calculation was performed with only the pure first order terms, effectively setting $I_{\lambda\lambda}(\cdot) = 0$ in (4.12), the discrepancy is considerably larger, with relative errors of up to 2% for larger assumed credit spreads. We conclude that, for riskier credits, the inclusion of the most important second order terms is advisable.

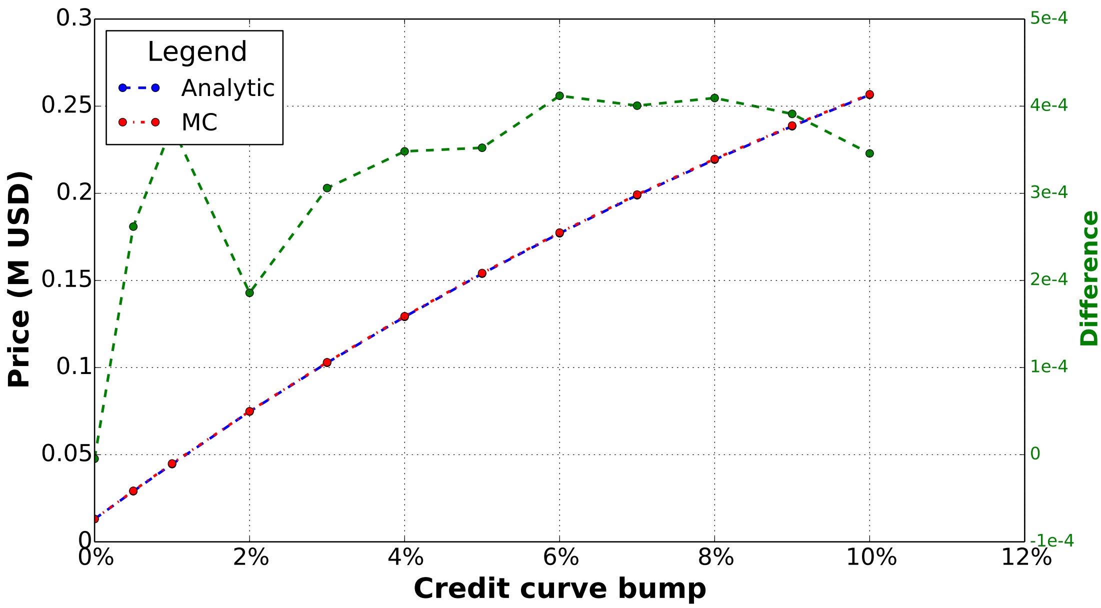

Figure 5: Dependence of cost of protection for quanto CDS on credit curve bump.

5.2 Risk Analysis

5.2.1 Quanto protection

In the following we make use of the methodology set out by Turfus (2018) for quantifying the model risk resulting from model parameter uncertainty, considering in particular here uncertainty in the values of the credit-FX correlation and/or the jump level to be used in modelling the prices of quanto CDS. The important point to consider is that in varying the model parameter inputs we should as far as possible maintain consistency with all available market data.

We note first that, assuming the correlation to be zero and the jump k to be a known value, equating (4.12) with (4.5) allows us to determine the foreign currency fair premium c. Alternatively, since this fair premium will often be quoted on the market (or estimated by analogy with other names quoted on both domestic and foreign currency CDS markets), we can think of the calibration condition of fitting model prices to the observed market value(s) as allowing an implied jump value k to be inferred.

Let us consider now introducing a *non-zero* value for $\rho_{\lambda z}$ into a deterministic model while at the same time maintaining the calibration condition. To do so, it will prove necessary to modify the assumed k value. To satisfy this constraint, we can define implicitly for each T a function $f_{T,k}:(-1,1)\to(-1,\infty)$ such that the implied value for k is $f_{T,k}(\rho_{\lambda z})$. Considering in particular small perturbations to the zero correlation state, we find that $f'_{T,k}(0)$ must satisfy

$$\left.\left(\frac{\partial V^{\mathrm{CDS}}_{\mathrm{protection}}}{\partial\rho_{\lambda z}}-\frac{\partial V_{\mathrm{coupon}}}{\partial\rho_{\lambda z}}\right)\right|_{\rho_{\lambda z}=0}+\left(\frac{\partial V^{\mathrm{CDS}}_{\mathrm{protection}}}{\partial k}-\frac{\partial V_{\mathrm{coupon}}}{\partial k}\right)f'_{T,k}(0)=0, \tag{5.1}$$

The differentiation is straightforward albeit tedious. Our task is simplified however if we ignore the (relatively small) impact of k through the drift in the FX rate (see (2.9)) in favour of its more direct impact through the jump and consider only first order effects by setting $I_{\lambda\lambda}(\cdot)=0$, whence we obtain

$$f'_{T,k}(0)\approx-(1+k)\frac{\int_0^T M(v)B(0,v)\overline{\lambda}(v)\left(\frac{\partial I_{\lambda z}(0,v)}{\partial\rho_{\lambda z}}-(1+k)\int_0^v\overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial\rho_{\lambda z}}du\right)dv}{\int_0^T M(v)B(0,v)\overline{\lambda}(v)dv} \tag{5.2}$$

We now look to consider the impact of correlation on the PV of a quanto CDS where a cap is imposed, which impact is given by the difference between (4.11) and (4.5). We note in the first instance that V_{coupon} is by assumption at all times identical to $V^{\text{CDS}}_{\text{protection}}$,[4] so it suffices to focus on the "exotic" component of the trade, namely

$$\Delta V_{\text{protection}} = V_{\text{protection}} - V^{\text{CDS}}_{\text{protection}} \tag{5.3}$$

We take the proposed FX-credit correlation adjustment to be used to "correct" the price obtained from the deterministic model to be $\Delta\rho_{\lambda z}$. It is then a straightforward matter to calculate the total derivative of (5.3) w.r.t. the correlation and multiply by the the correlation adjustment to obtain our approximations to the price adjustment. Again considering only the direct impact of k through the jump and first order effects in (4.11), we conclude:

$$\begin{aligned}
\text{Adjustment} &\approx -\Delta\rho_{\lambda z}\int_0^T M(v)B(0,v)\overline{\lambda}(v)\Bigg(\frac{f'_{T,k}(0)}{1+k} + \frac{\partial I_{\lambda z}(0,v)}{\partial\rho_{\lambda z}} - \int_0^v \overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial\rho_{\lambda z}}du \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left(1+k+\frac{kD}{\sqrt{I_{zz}(0,v)}}\right)\Bigg)N(d_{1,0}(v))\,dv \\
&\approx k\Delta\rho_{\lambda z}\int_0^T M(v)B(0,v)\overline{\lambda}(v)\frac{N'(-d_{1,0}(v))}{\sqrt{I_{zz}(0,v)}}\int_0^v \overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial\rho_{\lambda z}}du\,dv
\end{aligned} \tag{5.4}$$

where $d_{1,0}(\cdot)$ is obtained from $d_1(\cdot)$ by setting $I_{\lambda z}(.) \equiv 0$ and the operator D denotes differentiation of the function $N(\cdot)$ w.r.t. its argument. We see that the leading order impact of the assumed FX-credit correlation on the capped protection amount is effectively cancelled out by the recalibration which would have to take place for the model consistently to reflect the foreign currency CDS market prices. The residual adjustment occurs essentially because the moneyness parameter $d_1(\cdot)$ in (4.11) is impacted by the correlation assumption. Note that the adjustment goes to zero when $K \to \infty$ and/or when $k = 0$ is assumed. Since k and $\Delta\rho_{\lambda z}$ will in general both be negative, the adjustment will be such as to *increase* the price of protection; in other words, deterministic models are liable to overestimate the influence of the cap. Having said that, the size of the adjustment to the value of protection is likely to be extremely small, around 10^{-4} in relative terms.

Alternatively, if the foreign currency fair CDS premium c is *not* known, it will not be possible to impose the associated calibration condition, whence $f_{T,k}(\cdot)$ must effectively be taken to be zero. In this case, the discrepancy resulting from assuming an incorrect value for the FX-credit correlation will be obtained by a simple derivative of (4.11), there being no impact on the value of the coupon leg. We obtain straightforwardly, using the same approximation as previously,

$$\begin{aligned}
\text{Adjustment} &\approx \Delta\rho_{\lambda z}\int_0^T M(v)B(0,v)\overline{\lambda}(v)\Bigg(\frac{\partial I_{\lambda z}(0,v)}{\partial\rho_{\lambda z}} - \int_0^v \overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial\rho_{\lambda z}}du \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left(1+k-\frac{kD}{\sqrt{I_{zz}(0,v)}}\right)\Bigg)N(-d_{1,0}(v))\,dv.
\end{aligned} \tag{5.5}$$

Obviously the discrepancy in this case will be considerably larger and will furthermore not be diminished by the cap being out of the money.

We comment briefly here on the possible impact of stochastic *domestic* interest rates. The impact of correlation with the credit default intensity will largely be mitigated by recalibration to the domestic currency CDS market. Ehlers & Schönbucher (2006) show that the impact of correlation between the two rates has little impact. Correlation with FX will impact in a similar way on the value of both the coupon leg and the protection leg of a quanto CDS. So, provided the quanto CDS is close to the money, we would not expect the overall PV impact to be very significant. In other words, the expressions presented above represent the most important model parameter risk for quanto CDS pricing with capped losses.

[4] Of course, if the CDS under consideration is *not* at the money, the coupon leg will not match the protection leg; but as the impact of correlation on PV will be only through discount factor effects which we have already chosen to ignore, we can consistently take the CDS to be correctly priced irrespective of correlation and moneyness.

5.2.2 Quanto loss cap

The analysis for the quanto loss cap case based on (4.14) is similar. In this case, there is no impact of correlation or jump-at-default in the absence of a loss cap since the trade is then no longer quanto. If the foreign currency CDS rates are known and a unique value of $\rho_{\lambda z}$ can be chosen for given k, the model uncertainty will be small and given by

$$\text{Adjustment} \approx \Delta\rho_{\lambda z} K_F k(1+k) \int_0^T F(v)B(0,v)\overline{\lambda}(v)e^{-k\int_0^v \overline{\lambda}(s)ds} \frac{N'(-d^*_{1,0}(v))}{\sqrt{I_{zz}(0,v)}} \int_0^v \overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial \rho_{\lambda z}} du\, dv, \tag{5.6}$$

where $d^*_{1,0}(\cdot)$ is obtained from $d^*_1(\cdot)$ by setting $I_{\lambda z}(.) \equiv 0$ Alternatively, if the foreign currency fair CDS premium c is *not* known, we propose

$$\text{Adjustment} \approx \Delta\rho_{\lambda z} K_F (1+k) \int_0^T F(v)B(0,v)\overline{\lambda}(v)e^{-k\int_0^v \overline{\lambda}(s)ds} \left(\frac{\partial I_{\lambda z}(0,v)}{\partial \rho_{\lambda z}} - \int_0^v \overline{\lambda}(u)\phi_\lambda(u,v)\frac{\partial I_{\lambda z}(0,u)}{\partial \rho_{\lambda z}} du \left(1+k-\frac{kD}{\sqrt{I_{zz}(0,v)}}\right)\right) N(-d^*_{1,0}(v))\, dv. \tag{5.7}$$

This adjustment will be more significant although, unlike in the quanto protection case considered above, it will vanish in the limit as $K_F \to \infty$, since the trade is no longer quanto in that limit.

6 Conclusions

Approximate pricing formulae have been derived for CDS with the protection payment capped in terms of a currency different from that in which the debt is denominated, under the assumption that there may be a jump in the FX rate at the time of default. The (foreign) interest rate model is assumed to be of Hull-White type and the credit short-rate model to be Black-Karasinski, thus ensuring positive intensities. The formulae derived above are asymptotically valid to first order in the typical size of deviations of interest rates and default intensities from their respective forward rates. The most important second order effects are also included. The stochastic nature of the credit intensity is seen to have significant impact, but mainly through its correlation with the FX process; the stochastic foreign interest rate has no impact.

Comparisons of the derived quanto CDS formula with Monte Carlo simulations in the absence of a cap were found to be favourable. A model risk analysis concluded that most of the impact of wrongly ignoring correlation in pricing quanto CDS with capped loss payments can be straightforwardly mitigated by choosing a jump value k which allows vanilla CDS in the foreign currency market to be priced correctly. Explicit formulae were derived giving approximate expressions for the (small) impact of correlation which *cannot* be compensated by recalibration, in other words the "model uncertainty" which results either from not knowing the correlation or from using a model which is not capable of taking it into account. Formula were also derived for the larger impact of correlation uncertainty in the event that no foreign currency CDS market data are available.

A Derivation of Protection Leg Value

We consider the calculation of the perturbation series representation (4.8) of $Q(x,y,z,t)$ which solves (4.7) to first order subject to $Q(x,y,z,T) = 0$. To that end let us write the forcing function in (4.7) as

$$\lambda(y,t)P_{\text{def}}(z,t) = \sum_{j=0}^{\infty} P_j(y,z,t) \tag{A.1}$$

with $P_j(\cdot) = \mathcal{O}(\epsilon_\lambda^j)$. In particular

$$\begin{aligned} P_0(y,z,t) &= \overline{\lambda}(t)\mathcal{E}_y(y_t)P_{\mathrm{def}}(z,t), \\ P_1(y,z,t) &= 0, &\text{(A.2)} \\ P_2(y,z,t) &= \lambda_2^*(t)\mathcal{E}_y(y_t)P_{\mathrm{def}}(z,t). &\text{(A.3)} \end{aligned}$$

Gathering together all leading order terms, we obtain

$$\begin{aligned} Q_{0,0}(x,y,z,t) &= \int_t^T B(t,v) \iiint_{\mathbb{R}^3} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)P_0(\eta,\zeta,v)\,d\xi\,d\eta\,d\zeta\,dv \\ &= \int_t^T \frac{B(t,v)\overline{\lambda}(v)}{\sqrt{I_{zz}(t,v)}}\mathcal{E}_y(\phi_\lambda(t,v)y_t)\int_{\mathbb{R}} N'\left(\frac{\zeta - z - I_{\lambda z}(t,v)}{\sqrt{I_{zz}(t,v)}}\right)P_{\mathrm{def}}(\zeta,v)d\zeta\,dv \\ &= \int_t^T B(t,v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t,v)y_t)(M(v)\mathcal{E}_z(z_t + I_{\lambda z}(t,v))N(-d_1(z,t,v)) + KN(d_2(z,t,v)))\,dv, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} M(v) &:= (1-R)(1+k)F(v)e^{-k\int_0^v \overline{\lambda}(s)ds}, &\text{(A.5)} \\ d_2(z,t,v) &:= \frac{\ln M(v) - \ln K + z - \frac{1}{2}I_{zz}(0,v) + I_{\lambda z}(t,v)}{\sqrt{I_{zz}(t,v)}}, &\text{(A.6)} \\ d_1(z,t,v) &:= d_2(z,t,v) + \sqrt{I_{zz}(t,v)}. &\text{(A.7)} \end{aligned}$$

Likewise

$$\begin{aligned} Q_{1,0}(x,y,z,t) &= \int_t^T B(t,v) \iiint_{\mathbb{R}^3} G_{1,0}(x,y,z,t;\xi,\eta,\zeta,v)P_0(\eta,\zeta,v)\,d\xi\,d\eta\,d\zeta\,dv \\ &= -\int_t^T B(t,v) \iiint_{\mathbb{R}^3} B^*(t,v)(x - I_{rz}(0,t))\frac{\partial}{\partial z} \\ &\qquad G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)P_0(\eta,\zeta,v)\,d\xi\,d\eta\,d\zeta\,dv \\ &= -\int_t^T \frac{B(t,v)\overline{\lambda}(v)}{\sqrt{I_{zz}(t,v)}}\mathcal{E}_y(\phi_\lambda(t,v)y_t)B^*(t,v)(x - I_{rz}(0,t)) \\ &\qquad \frac{\partial}{\partial z}\int_{\mathbb{R}} N'\left(\frac{\zeta - z - I_{\lambda z}(t,v)}{\sqrt{I_{zz}(t,v)}}\right)P_{\mathrm{def}}(\zeta,v)\,d\zeta\,dv \\ &= -\int_t^T B(t,v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t,v)y_t)M(v)\mathcal{E}_z(z_t + I_{\lambda z}(t,v))B^*(t,v)(x - I_{rz}(0,t)) \\ &\qquad N(-d_1(z,t,v))\,dv. \end{aligned} \tag{A.8}$$

Similarly we deduce

$$
\begin{aligned}
Q_{0,1}(x,y,z,t) &= \int_t^T B(t,v) \iiint_{\mathbb{R}^3} G_{0,1}(x,y,z,t;\xi,\eta,\zeta,v) P_0(\eta,\zeta,v)\, d\xi\, d\eta\, d\zeta\, dv \\
&\sim -\int_t^T B(t,v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t,v)y_t)M(v)\mathcal{E}_z(z_t+I_{\lambda z}(t,v))\left(1+k+k\frac{\partial}{\partial z}\right)N(-d_1(z,t,v)) \\
&\qquad \int_t^v \overline{\lambda}(u)\left(\mathcal{E}_y(\phi_\lambda(t,u)y_t)e^{\phi_\lambda(u,v)(I_{\lambda\lambda}(t,u)+I_{\lambda z}(t,u))}-1\right) du\, dv \\
&\quad - K\int_t^T B(t,v)\overline{\lambda}(v)\mathcal{E}_y(\phi_\lambda(t,v)y_t)\left(1+k\frac{\partial}{\partial z}\right)N(d_2(z,t,v)) \\
&\qquad \int_t^v \overline{\lambda}(u)\left(\mathcal{E}_y(\phi_\lambda(t,u)y_t)e^{\phi_\lambda(u,v)I_{\lambda\lambda}(t,u)}-1\right) du\, dv. \qquad \text{(A.9)}
\end{aligned}
$$

For completeness, we consider also the influence of $P_2(\cdot)$, denoting this by

$$
\begin{aligned}
Q_{0,2}^{(1)}(x,y,z,t) &= \int_t^T B(t,v) \iiint_{\mathbb{R}^3} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) P_2(\eta,\zeta,v)\, d\xi\, d\eta\, d\zeta\, dv \\
&= \int_t^T B(t,v)\gamma_{0,2}^*(v)\mathcal{E}_y(\phi_\lambda(t,v)y_t) \\
&\qquad (M(v)\mathcal{E}_z(z_t+I_{\lambda z}(t,v))N(-d_1(z,t,v))+KN(d_2(z,t,v)))\, dv. \qquad \text{(A.10)}
\end{aligned}
$$

Although this contribution is formally second order in ϵ_λ, unlike the terms arising from application of $G_{0,2}(\cdot)$ which are third order in $\overline{\lambda}(\cdot)$, it is only second order in $\overline{\lambda}(\cdot)$. Since the smallness of ϵ_λ is largely driven by the smallness of $\overline{\lambda}(\cdot)$, this term is likely to be considerably more important than the $G_{0,2}(\cdot)$ terms. A further justification for its inclusion is that it provides terms matching those arising in the above first order calculation.

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